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Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations

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ABSTRACT

A systematic investigation to derive Lie point symmetries to time fractional generalized Burgers as well as Korteweg–de Vries equations is presented. Using the obtained Lie point symmetries we have shown that each of them has been transformed into a nonlinear ordinary differential equation of fractional order with a new independent variable. The derivative corresponding to time fractional in the reduced equation is usually known as the Erdélyi–Kober fractional derivative.

1. Introduction

It is well known that the Lie group of transformations theory, originally advocated by the Norwegian mathematician Sophus Lie in the beginning of the 19th century, plays a significant role in the analysis of differential equations [1–6]. Now this transformation group theory is popularly known as Lie symmetry analysis in the literature. The basic idea of the Lie symmetry analysis is the consideration of the tangent structural equations under one or several parameter transformation groups in conjunction with the system of differential equations. It is appropriate to mention here that for nonlinear partial differential equations (PDEs) with two independent variables exhibiting solitons, the Lie symmetry analysis not only helps to study their group theoretical properties but also to derive several mathematical characteristics related with their complete integrability [3,4,7,8]. Recently, this method has been successfully extended to discrete systems exhibiting solitons governed by nonlinear partial differential–difference equations (PDΔEs) with two independent variables and pure difference equations or lattice equations ($\Delta\Delta$s) with one or more independent variables including mappings or lattice equations and demonstrated how it provides an efficient tool to derive different mathematical properties related with its complete integrability [9–11]. It has also been illustrated that how this Lie symmetry analysis can be effectively used to find exact solutions of both ODEs and PDEs apart from PDΔEs and $\Delta\Delta$s.

In recent years, the study of fractional ODEs and PDEs has attracted much attention due to an exact description of nonlinear phenomena in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science [12–15]. In reality, a physical phenomenon may depend not only on the time instant but also on the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [14,15]. The time fractional PDEs are obtained by replacing the integer order time derivative in PDEs by the fractional derivative of order $\alpha > 0$. Given a fractional differential equation (FDE), there exists no well-defined method to analyze them systematically. Much efforts have been spent in recent years to develop techniques to deal with FDEs. As a consequence, several ad hoc methods such as the Finite difference method [16], Adomian decomposition method [17], Variational iteration method [18], Homotopy perturbation method [19], Laplace transform method [20], Fourier transform method [14], Generalized differential transform method [21]

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and other numerical schemes [22,23] have been formulated. The primary objective of this article is to investigate whether or not the Lie symmetry analysis be useful in the analysis of fractional dynamical systems. To the best of our knowledge, the Lie symmetry analysis has not been widely applied for studying the invariance properties of FDEs. However several research groups have been engaged in this direction [24–27]. For example, the authors of Ref. [26] have considered the time fractional linear wave-diffusion equation and obtained a group of dilation. Using the dilation symmetries they have constructed scale invariant solutions. In Ref. [24], the authors have made an attempt to extend the Lie symmetry analysis to FDEs (see also [25]).

In this article, we consider the following time fractional PDEs

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xx} + Au^p u_x, \tag{1}$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u_{xxx} + Au^p u_x, \tag{2}$$

where $0 < \alpha \leq 1, p > 0$, which occur in different contexts in mathematical physics; here the time fractional derivative leads to sub-diffusion and sub-dispersion respectively and extend the Lie symmetry analysis to derive their infinitesimals. Using the infinitesimals, we find their Lie algebra and show that each of them can be transformed into a nonlinear ODE of fractional order.

We would like to mention that there exists no unique notion to define the concept of fractional derivative [14,15]. In the literature, several definitions of the fractional derivative [15] such as the Grunwald–Letnikov, the Riemann–Liouville, the Weyl, the Caputo, the Riesz, and the Miller and Ross [14] have been adopted by different researchers. In this article, we follow the definition of the Riemann–Liouville fractional derivative, which is defined by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^m u}{\partial t^m}, & \alpha = m \in \mathbb{N}; \\ \dfrac{1}{\Gamma(m-\alpha)} \dfrac{\partial^m}{\partial t^m} \displaystyle\int_0^t \frac{u(\tau, x)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m,\ m \in \mathbb{N}. \end{cases} \tag{3}$$

2. Symmetry analysis of fractional partial differential equations

To be self contained we present below brief details of the Lie symmetry analysis for FPDEs with two independent variables. Consider a scalar time fractional PDE having the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = F(x, t, u, u_x, u_{xx}, u_{xxx}, \ldots), \quad \alpha > 0 \tag{4}$$

where subscripts denote partial derivatives. Let us assume that the above FPDE, (4), is invariant under a one parameter (ϵ) continuous transformations

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(x, t, u) + O(\epsilon^2), \\ \bar{x} &= x + \epsilon\xi(x, t, u) + O(\epsilon^2), \\ \bar{u} &= u + \epsilon\eta(x, t, u) + O(\epsilon^2), \\ \frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\zeta_\alpha^0 + O(\epsilon^2), \\ \frac{\partial \bar{u}}{\partial \bar{x}} &= \frac{\partial u}{\partial x} + \epsilon\zeta_1^1 + O(\epsilon^2), \\ \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\zeta_2^1 + O(\epsilon^2), \\ \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \epsilon\zeta_3^1 + O(\epsilon^2), \\ &\vdots \end{aligned} \tag{5}$$

where τ, ξ and η are infinitesimals and ζ_1^1, ζ_2^1, ζ_3^1 and ζ_α^0 are extended infinitesimals of orders 1, 2, 3 and α respectively. The explicit expression for ζ_1^1, ζ_2^1, and ζ_3^1 are

$$\begin{aligned} \zeta_1^1 &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \\ \zeta_2^1 &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (n_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 \\ &\quad - \tau_{uu}u_x^2 u_t + (\eta_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x, \end{aligned}$$

$$\begin{aligned}\zeta_3^1 \;=\; & \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t + 3(\eta_{xuu} - \xi_{xxu})u_x^2 - 3\tau_{xxu}u_xu_t \\ & + (\eta_{uuu} - 3\xi_{xuu})u_x^3 + 3(\eta_{xu} - \xi_{xx})u_{xx} - 3\tau_{xx}u_{xt} - 3\tau_{xuu}u_x^2u_t + 3(\eta_{uu} - 3\xi_{xu})u_xu_{xx} \\ & - 3\tau_{xu}u_tu_{xx} - 6\tau_{xu}u_{xt}u_x - 3\tau_xu_{xxt} + (\eta_u - 3\xi_x)u_{xxx} - \xi_{xxx}u_x^4 - 6\xi_{uu}u_x^2u_{xx} - 3\tau_{uu}u_x^2u_{tx} \\ & - \tau_{uuu}u_x^3u_t - 3\xi_uu_{xx}^2 - 3\tau_uu_{xxt}u_x - 3\tau_uu_{xt}u_{xx} - 3\tau_{uu}u_{xx}u_xu_t - 4\xi_uu_{xxx}u_x - \tau_uu_{xxx}u_t,\end{aligned}$$

with infinitesimal generator $X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}$. Since the lower limit of the integral in (3) is fixed and, therefore it should be invariant with respect to the transformations (5). Such invariance condition arrives at

$$\tau(t, x, u)\,|_{t=0} = 0. \tag{6}$$

The αth extended infinitesimal related to Riemann–Liouville fractional time derivative with (6) reads (see [24,25])

$$\zeta_\alpha^0 = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u). \tag{7}$$

Here the operator D_t^α denotes the total fractional derivative operator. We would like to recall the generalized Leibnitz rule [14,28] given by

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^{\alpha-n}f(t)D_t^n g(t), \quad \alpha > 0, \tag{8}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}.$$

Using the Leibnitz rule (8), the above Eq. (7) can be written as

$$\zeta_\alpha^0 = D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau)D_t^{\alpha-n}(u). \tag{9}$$

Also we would like to recall the generalization of the well known chain rule for composite function [29] given by

$$\frac{d^m g(y(t))}{dt^m} = \sum_{k=0}^{m}\sum_{r=0}^{k}\binom{k}{r}\frac{1}{k!}[-y(t)]^r\frac{d^m}{dt^m}[(y(t))^{k-r}]\frac{d^k g(y)}{dy^k}. \tag{10}$$

Further, using the chain rule (10) along with the generalized Leibnitz rule (8) with $f(t) = 1$, one can written the first term $D_t^\alpha(\eta)$ in the RHS of (9) as

$$D_t^\alpha(\eta) = \frac{\partial^\alpha\eta}{\partial t^\alpha} + \eta_u\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}D_t^{\alpha-n}(u) + \mu, \tag{11}$$

where

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

As a consequence the αth extended infinitesimal given in (9) becomes

$$\begin{aligned}\zeta_\alpha^0 \;=\; & \frac{\partial^\alpha\eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ & - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x).\end{aligned} \tag{12}$$

For the invariance of FPDE (4) under transformations (5), we have

$$\frac{\partial^\alpha\bar{u}(\bar{x},\bar{t})}{\partial\bar{t}^\alpha} = F(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}}, \ldots) \tag{13}$$

for any solution $u = u(x, t)$ of FPDE (4). Expanding (13) about $\epsilon = 0$ and making use of infinitesimals and their extensions (5) and equating the coefficients of ϵ, and neglecting the terms of higher powers of ϵ, we obtain

$$\left[\zeta_\alpha^0 - \xi\frac{\partial F}{\partial x} - \tau\frac{\partial F}{\partial t} - \eta\frac{\partial F}{\partial u} - \zeta_1^1\frac{\partial F}{\partial u_x} - \zeta_2^1\frac{\partial F}{\partial u_{xx}} - \zeta_3^1\frac{\partial F}{\partial u_{xxx}} - \cdots\right]_{(4)} = 0 \tag{14}$$

which is known as the invariant equation of FPDE (4). Now solving the invariant equation (14), one can determine τ, ξ, η explicitly. It is appropriate to mention that the expression for μ given in (11) vanishes when the infinitesimal η is linear in u.

Definition 2.1. A solution $u = \theta(x, t)$ is said to be an invariant solution of FPDE (4) if and only if

(i) $u = \theta(x, t)$ is an invariant surface, i.e. $X\theta = 0 \Longrightarrow \left(\tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\right)\theta = 0$.
(ii) $u = \theta(x, t)$ satisfies FPDE (4).

3. Time fractional generalized Burgers equation

The symmetry analysis of the Burgers equation ($\alpha = 1, p = 1$) is well known (see [2,4]). Let us assume that the time fractional generalized Burgers equation, (1), is invariant under a one parameter transformations (5), and so the transformed equation reads

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} = \bar{u}_{\bar{x}\bar{x}} + A\bar{u}^p \bar{u}_{\bar{x}} \tag{15}$$

provided $u = u(x, t)$ satisfies (1). Making use of transformations (5) in (15) we obtain the following invariant equation

$$[\zeta_\alpha^0 - \zeta_2^1 - Au^p\zeta_1^1 - p\eta u^{p-1}u_x]|_{(1)} = 0 \tag{16}$$

which depend on variables $u_x, u_{xx}, u_{xt}, u_t, \ldots$ and $D_t^{\alpha-n}u, D_t^{\alpha-n}u_x$ for $n = 1, 2, \ldots$ which are considered to be independent. Such a structure of (16) allows one to reduce it to a system of infinitely many linear FDEs. Substituting the expressions for ζ_1^1, ζ_2^1 and ζ_α^0 given in (5) and (12) into (16) and equating various powers of derivatives of u to zero we obtain an over determined system of linear equations. They are

$$\begin{aligned}
&\xi_u = \xi_t = \tau_u = \tau_x = \eta_{uu} = 0,\\
&\binom{\alpha}{n}\partial_t^n(\eta_u) - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0 \quad \text{for } n = 1, 2, \ldots,\\
&\xi''(x) - Au^p\alpha\tau'(t) - 2\eta_{xu} + Au^p\xi'(x) - Ap\eta u^{p-1} = 0,\\
&2\xi'(x) - \alpha\tau'(t) = 0,\\
&\partial_t^\alpha(\eta) - u\partial_t^\alpha(\eta_u) - \eta_{xx} - Au^p\eta_x = 0.
\end{aligned} \tag{17}$$

Solving system (17) consistently, we obtain the explicit form of infinitesimals

$$\xi = a_0x + a_1, \qquad \tau = \frac{2a_0t}{\alpha}, \qquad \eta = \frac{-a_0u}{p} \tag{18}$$

where a_0 and a_1 are arbitrary constants. Hence the infinitesimal operator becomes

$$X = (a_0x + a_1)\frac{\partial}{\partial x} + \frac{2a_0t}{\alpha}\frac{\partial}{\partial t} - \frac{a_0u}{p}\frac{\partial}{\partial u}$$

and so the underlying Lie algebra of time fractional generalized Burgers equation is two dimensional with basis ($X_1 = \frac{\partial}{\partial x}, X_2 = x\frac{\partial}{\partial x} + \frac{2t}{\alpha}\frac{\partial}{\partial t} - \frac{u}{p}\frac{\partial}{\partial u}$).

The similarity variable and similarity transformation corresponding to the infinitesimal generator X_2 can be obtained by solving the associated characteristic equation given by

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{-pdu}{u}$$

which respectively take the following form

$$z = xt^{\frac{-\alpha}{2}}, \quad u = t^{\frac{-\alpha}{2p}}f(z). \tag{19}$$

Theorem 3.1. *The similarity transformation $u(x, t) = t^{\frac{-\alpha}{2p}}f(z)$ along with the similarity variable $z = xt^{\frac{-\alpha}{2}}$ reduces the time fractional generalized Burgers equation (1) to the nonlinear ordinary differential equation of fractional order of the form*

$$\left(P_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p}-\alpha,\alpha}f\right)(z) = \frac{d^2f}{dz^2} + Af^p\frac{df}{dz} \tag{20}$$

with the Erdélyi–Kober fractional differential operator [30]

$$\left(P_\delta^{\tau,\alpha}g\right)(z) := \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\delta}z\frac{d}{dz}\right)\left(K_\delta^{\tau+\alpha,m-\alpha}g\right)(z), \quad z > 0,\ \delta > 0,\ \alpha > 0 \tag{21}$$

$$m = \begin{cases}[\alpha] + 1, & \alpha \notin \mathbb{N}\\ \alpha, & \alpha \in \mathbb{N}\end{cases}$$

where

$$\left(K_\delta^{\tau,\alpha}g\right)(z) := \begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty (v-1)^{\alpha-1}v^{-(\tau+\alpha)}g\left(zv^{\frac{1}{\delta}}\right)dv, & \alpha > 0;\\ g(z), & \alpha = 0\end{cases} \tag{22}$$

is the Erdélyi–Kober fractional integral operator.

Proof. Let $n-1<\alpha<n, n=1,2,3,\ldots$. Then the Riemann–Liouville fractional derivative for the similarity transformation (19) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}s^{-\frac{\alpha}{2p}}f\left(xs^{-\frac{\alpha}{2}}\right)ds\right].$$

Let $\upsilon=\frac{t}{s}$. Then the above equation can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2p}}\frac{1}{\Gamma(n-\alpha)}\int_1^\infty (\upsilon-1)^{n-\alpha-1}\upsilon^{-\left(n-\alpha-\frac{\alpha}{2p}+1\right)}f\left(z\upsilon^{\frac{\alpha}{2}}\right)d\upsilon\right].$$

Following the definition of the Erdélyi–Kober fractional integral operator given in (22), we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2p}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right]. \tag{23}$$

In order to simplify the RHS of (23), we consider the relation ($z=xt^{-\frac{\alpha}{2}}$, $\phi\in C^1(0,\infty)$)

$$t\frac{\partial}{\partial t}\phi(z)=tx\left(-\frac{\alpha}{2}\right)t^{-\frac{\alpha}{2}-1}\phi'(z)=-\frac{\alpha}{2}z\frac{d}{dz}\phi(z)$$

and so, we get

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2p}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right]&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[\frac{\partial}{\partial t}\left(t^{n-\alpha-\frac{\alpha}{2p}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right)\right]\\&=\frac{\partial^{n-1}}{\partial t^{n-1}}\left[t^{n-\alpha-\frac{\alpha}{2p}-1}\left(n-\alpha-\frac{\alpha}{2p}-\frac{\alpha}{2}z\frac{d}{dz}\right)\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right].\end{aligned}$$

Repeating the similar procedure for $n-1$ times, we have

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2p}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right]=t^{-\alpha-\frac{\alpha}{2p}}\prod_{j=0}^{n-1}\left(1-\frac{\alpha}{2p}-\alpha+j-\frac{\alpha}{2}z\frac{d}{dz}\right)\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z).$$

Now using the definition of the Erdélyi–Kober fractional differential operator given in (21), the above equation can be written as

$$\frac{\partial^n}{\partial t^n}\left[t^{n-\alpha-\frac{\alpha}{2p}}\left(K_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p},n-\alpha}f\right)(z)\right]=t^{-\alpha-\frac{\alpha}{2p}}\left(P_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p}-\alpha,\alpha}f\right)(z).$$

Thus we obtain an expression for the time fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{-\alpha-\frac{\alpha}{2p}}\left(P_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p}-\alpha,\alpha}f\right)(z). \tag{24}$$

Continuing further we find that the time fractional generalized Burgers equation (1) reduces into an ordinary differential equation of fractional order

$$\left(P_{\frac{2}{\alpha}}^{1-\frac{\alpha}{2p}-\alpha,\alpha}f\right)(z)=\frac{d^2f}{dz^2}+Af^p\frac{df}{dz}.\quad\square \tag{25}$$

The above nonlinear differential equation of fractional order is not solvable, in general. However, when $A=0$ it possesses two independent solutions [26]. When $p=1$ and $A\neq 0$ the construction of a particular solution of Eq. (25) is under investigation.

4. Time fractional generalized Korteweg–de Vries equation

In this section, we study the invariance properties of the time fractional generalized Korteweg–de Vries equation. Let us assume that the time fractional generalized Korteweg–de Vries equation, (2), is invariant under a one parameter transformations (5), and so the transformed equation reads

$$\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha}=\bar{u}_{\bar{x}\bar{x}\bar{x}}+A\bar{u}^p\bar{u}_{\bar{x}} \tag{26}$$

provided $u=u(x,t)$ satisfies (2).

Making use of transformations (5) in (26), we obtain the following invariant equation

$$\left[\zeta_\alpha^0 - \zeta_3^1 - Au^p\zeta_1^1 - Ap\eta u^{p-1}u_x\right]_{(2)} = 0. \tag{27}$$

Substituting the expressions for ζ_1^1, ζ_3^1 and ζ_α^0 given in (5) and (12) into (27) and equating various powers of derivatives of u to zero we obtain an over determined system of linear equations. They are

$$\begin{aligned}
&\xi_u = \xi_t = \tau_u = \tau_x = \xi_{xxx} = \eta_{uu} = 0\\
&\binom{\alpha}{n}\partial_t^n(\eta_u) - \binom{\alpha}{n+1}D_t^{n+1}(\tau) = 0 \quad \text{for } n = 1, 2, \ldots\\
&Au^p\xi'(x) - Ap\eta u^{p-1} - 3\eta_{xxu} - \alpha Au^p\tau'(t) = 0\\
&3\xi''(x) - \eta_{xu} = 0\\
&3\xi'(x) - \alpha\tau'(t) = 0\\
&\partial_t^\alpha(\eta) - u\partial_t^\alpha(\eta_u) - \eta_{xxx} - Au^p\eta_x = 0.
\end{aligned} \tag{28}$$

Solving the system of equations (28) consistently, we obtain the explicit form of infinitesimals

$$\xi = a_0x + a_1, \qquad \tau = \frac{3a_0t}{\alpha}, \qquad \eta = \frac{-2a_0u}{p} \tag{29}$$

where a_0 and a_1 are arbitrary constants. Hence the infinitesimal operator becomes

$$X = (a_0x + a_1)\frac{\partial}{\partial x} + \frac{3a_0t}{\alpha}\frac{\partial}{\partial t} - \frac{2a_0u}{p}\frac{\partial}{\partial u}$$

and so the underlying Lie algebra of time fractional generalized Korteweg–de Vries equation is two dimensional with basis $(X_1 = \frac{\partial}{\partial x}, X_2 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - \frac{2u}{p}\frac{\partial}{\partial u})$.

The similarity variable z and similarity transformation $f(z)$ corresponding to the infinitesimal generator X_2 can be obtained by solving the associated characteristic equation given by

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{-pdu}{2u}.$$

They are

$$u = t^{\frac{-2\alpha}{3p}}f(z), \quad z = xt^{\frac{-\alpha}{3}}. \tag{30}$$

By using the above similarity transformation (30) in (2), we find that the time fractional generalized Korteweg–de Vries equation is transformed into a nonlinear ODE of fractional order and hence the following theorem.

Theorem 4.1. *The transformation $u = t^{\frac{-2\alpha}{3p}}f(z), z = xt^{\frac{-\alpha}{3}}$ reduces the time fractional generalized Korteweg–de Vries equation (2) to the nonlinear ordinary differential equation of fractional order of the form*

$$\left(P_{\frac{3}{\alpha}}^{1-\frac{2\alpha}{3p}-\alpha,\alpha}f\right)(z) = \frac{d^3f}{dz^3} + Af^p\frac{df}{dz} \tag{31}$$

with the Erdélyi–Kober fractional differential operator

$$\left(P_\delta^{\tau,\alpha}g\right)(z) := \prod_{j=0}^{m-1}\left(\tau + j - \frac{1}{\delta}z\frac{d}{dz}\right)\left(K_\delta^{\tau+\alpha,m-\alpha}g\right)(z), \quad z > 0,\ \delta > 0,\ \alpha > 0 \tag{32}$$

$$m = \begin{cases}[\alpha] + 1, & \alpha \notin \mathbb{N}\\ \alpha, & \alpha \in \mathbb{N}\end{cases}$$

where

$$\left(K_\delta^{\tau,\alpha}g\right)(z) := \begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_1^\infty (u-1)^{\alpha-1}u^{-(\tau+\alpha)}g\left(zu^{\frac{1}{\delta}}\right)du, & \alpha > 0;\\ g(z), & \alpha = 0\end{cases} \tag{33}$$

is the Erdélyi–Kober fractional integral operator.

Proof. The proof is similar to Theorem 3.1. □

Here again the nonlinear differential equation of fractional order, (31), is not solvable, in general. However when $A = 0$ one can derive three independent solutions of it following the procedure given in [26].

5. Summary and discussion

In this article, an attempt is made to illustrate the application of Lie symmetry approach to study time fractional nonlinear partial differential equations. More precisely, we consider time fractional generalized Burgers as well as Korteweg–de Vries equations and derived their Lie point symmetries. The Lie symmetry analysis show that the underlying symmetry algebra of each of the equations is two dimensional. The reduction of dimension in the symmetry algebra is due to the fact that each of the time fractional equations is not invariant under time translation symmetry. Using the Lie point symmetries, we have shown that each of the equations can be transformed into a nonlinear ODE of fractional order which is not solvable as in the case of $\alpha = 1$ and $p = 1$. It is known that when $\alpha = 1$ and $p = 1$ the transformed equations can be integrated further leading to Painleve transcendental equations ensuring their integrability. It is not clear at the moment how to derive similar results in the case of time fractional PDEs with two independent variables.

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